Journal of Statistical Physics, Vol. 29, No. 4, 1982

Triviality of ϕ_4^4 and All That in a Hierarchical Model Approximation

K. Gawędzki[1,3] **and A. Kupiainen**[2]

Received June 25, 1982

The infrared behavior of lattice ϕ_d^4, $d \geqslant 4$, and dipole gases in $d \geqslant 1$ is rigorously shown to be Gaussian within the context of a hierarchical approximation. Methods are developed to generalize the result beyond this approximation.

KEY WORDS: Renormalization group; infrared fixed point; dipole gas; ϕ^4 theories on a lattice; hierarchical model; block spins.

1. INTRODUCTION

In this paper we develop some new methods allowing a nonperturbative control of the Kadanoff–Wilson renormalization group (RG). Our aim is to apply the RG to prove that the long-distance behavior of weakly coupled lattice ϕ^4 theories in dimensions $d \geqslant 4$ and dipole gases and the like in all dimensions is Gaussian. Here we will establish this claim in a hierarchical approximation to these models.

In previous papers the authors have used the RG to study certain massless perturbations of the free massless lattice field. Reference 1 studied the pressure of the $(\nabla\phi)^4$ model and Refs. 2 and 3 the correlation functions and RG trajectories of Hamiltonians in a simplified version of this model. However, that analysis was not completely satisfactory due to the boundedness of the block fluctuations we had to assume. In the present paper we use some techniques based on analyticity properties of the transformation to control this problem. In order to separate it from other ones we

[1] Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France.
[2] Helsinki University, Research Institute for Theoretical Physics, 00170 Helsinki, Finland.
[3] On leave of absence from the Department of Mathematical Methods of Physics, University, 00-682 Warsaw, Hoza 74, Poland.

work in a hierarchical approximation. To briefly recapitulate the motivation for the definition of the hierarchical models we recall[1] that the massless Gaussian lattice field ϕ can be decomposed in terms of (almost) identically distributed independent massive flucutation fields $\mathscr{Z}^n$ describing fluctuations on scale L^n:

$$\phi_x = \sum_{n=0}^{\infty} L^{-n(d-2)/2}(\mathscr{A}^n \mathscr{Z}^n)_{x/L^n} \tag{1.1}$$

where the kernels $\mathscr{A}^n$ on $L^{-n}\mathbb{Z}^d \times \mathbb{Z}^d$ have exponential decay uniformly in n. The hierarchical approximation is obtained by making $\mathscr{A}^n$ local and by representing the $\mathscr{Z}^n_x$'s in a block by one $\mathscr{Z}^n_{x_0}$. More precisely we write

$$\phi_x = \sum_{n=0}^{\infty} L^{-n(d-2)/2} A_{[L^{-n}x]} Z^n_{[L^{-n-1}x]} \tag{1.2}$$

where [ ] denotes the integral part. We take L even, $A_x = \pm 1$, and the average of A over the blocks of size L^d to be zero. Z^n's are assumed to be ultralocal: all Z^n_x's are independent with the same Gaussian distribution.

The approximations made above consist essentially of cutting off exponentially decaying tails of various kernels. In subsequent papers this error will be dealt with by a cluster expansion. Here we will only restrict ourselves to developing the analyticity techniques to deal with the unboundedness of the Z^n's. These will generalize in a natural way to the realistic models.

Our hierarchical model is closely related to the one introduced by Dyson.[4] The proofs will easily be adapted also to this model which was previously considered by Bleher and Sinai,[5,6] who proved the results for ϕ^4 in $d > 4$. Besides covering the interesting borderline case $d = 4$ our proof is considerably simpler than the quite involved one of Refs. 5 and 6.

Finally let us comment on the relation of the present results to those of Ref. 7. The infrared asymptotic freedom of weakly coupled lattice ϕ_4^4 (proven here for the hierarchical approximation) does not imply that all scaling limits of this theory are trivial, leaving the possibility of building such a limit around a strong coupling ultraviolet fixed point. This possibility is (almost) eliminated in Ref. 7 by means of correlation inequalities providing the information about the model in the whole range of coupling constants. Nevertheless extending our control of RG to theories like ϕ_4^4 with slow ($\sim$ inverse logarithm of the momentum) convergence to infrared Gaussian fixed point would be a big step to control similar situations in the ultraviolet region occurring, e.g., in the $d = 2$ nonlinear σ model or $d = 4$ non-Abelian gauge theory. There, however, the implication would be the existence of a nontrivial continuum limit.

The results of this paper appear in Section 2 and the main ideas involved are in Section 3.1. The subtleties of ϕ_4^4 can be found in Section 3.2. Reading the paper requires no previous knowledge of RG.

2. THE MODELS AND THE RESULTS

Most of this paper is devoted to the study of the following nonlinear transformation t:

$$\exp[-tv(\psi)] = \frac{\int d\mu(z)\exp\{-\frac{1}{2}L^d[v(\gamma\psi+z)+v(\gamma\psi-z)]\}}{\int d\mu(z)\exp[-L^d v(z)]} \tag{2.1}$$

where

$$d\mu(z) = \frac{1}{(2\pi)^{1/2}}\exp\left(-\frac{1}{2}z^2\right)dz.$$

γ is a parameter taking values $L^{-d/2}$, $d \geqslant 1$, or $L^{-(d-2)/2}$, $d \geqslant 4$. L is an even integer. The case $L^{-d/2}$ will be called the "dipole gas model" and the case $L^{-(d-2)/2}$ the "ϕ^4 model." The first is related to perturbations of the massless Gaussian which depend on ϕ through $\nabla\phi$ only, such as $\sum_{\mu,x}(\nabla_\mu\phi_x)^4$ (the anharmonic crystal) or $-\sum_x \int d\nu(\hat{n})\cos(\hat{n}\cdot\nabla\phi_x)$ (the dipole gas). The connection between these models and (2.1) is the following. From (1.1) we have

$$\nabla\phi_x = \sum_n L^{-nd/2}(\nabla\mathcal{C}^n\mathcal{Z}^n)_{x/L^n}. \tag{2.2}$$

We write a hierarchical approximation to this by taking as our basic field

$$\phi_x = \sum_{n=0}^{N-1} L^{-nd/2}A_{[L^{-n}x]}Z^n_{[L^{-n-1}x]} \tag{2.3}$$

where we have put the system in a box of size L^{Nd}. A will be chosen so that $A_x = \pm 1$ and $\sum_{[L^{-1}x]=y}A_x = 0$ for all y and that A is invariant under translations by L in each direction. The Z_x^n's are all independent Gaussians with covariance one. In the ϕ^4 case the basic field is given by (1.2). In both cases we introduce a potential $V(\phi)$ to be specified below and define the expectation

$$\langle - \rangle_V^N = \frac{1}{\mathcal{N}}\int \exp[-V(\phi)]\prod_{n=0}^{N-1}\prod_x d\mu(Z_x^n) \tag{2.4}$$

with $\mathcal{N}$ being the normalization. The RG transform TV of V is just the Z^0 integral

$$\exp[-TV(\psi)] = \frac{\int \exp[-V(\gamma\psi_{[\cdot/L]} + A\cdot Z^0_{[\cdot/L]})]\prod_x d\mu(Z_x^0)}{\int \exp[-V(A\cdot Z^0_{[\cdot/L]})]\prod_x d\mu(Z_x^0)} \tag{2.5}$$

where, basing on (2.3) or (1.2), we have written

$$\phi_x = \gamma\psi_{[x/L]} + A_x Z^0_{[x/L]} \tag{2.6}$$

γ being either $L^{-d/2}$ or $L^{-(d-2)/2}$.

The point of the hierarchical model is that for a local V, $V(\phi) = \sum_x v(\phi_x)$, T factorizes:

$$TV(\psi) = \sum_x tv(\psi_x) \tag{2.7}$$

with

$$\exp\left[-tv(\psi_x)\right] = \frac{\int \exp\left[-\sum_{[y/L]=x} v(\gamma\psi_x + A_y z)\right] d\mu(z)}{\int \exp\left[-\sum_{[y/L]=x} v(A_y z)\right] d\mu(z)} \tag{2.8}$$

With the use of $A_y = \pm 1$, $\sum_{[y/L]=x} A_y = 0$ and assumption that v is even, (2.8) becomes (2.1) (given the assumptions on v stated below, there is some abuse of notation (2.1) and (2.8): e^{-v} and e^{-tv} may vanish somewhere).

Our aim is to study the transformation t and to show that its iteration drives a very general class of v's to a "Gaussian fixed point." More precisely, for $\gamma = L^{-d/2}$ we wish to show that $t^n v \to \frac{1}{2}c(v)\phi^2$ (for all d) and for $\gamma = L^{-(d-2)/2}$, $t^n[v - \frac{1}{2}c(v)\phi^2] \to 0$ for a suitable "mass counterterm" $\frac{1}{2}c\phi^2$ (in the first case c corresponds to the wave function renormalization i.e. the dielectric constant). Moreover we wish to establish a sufficiently strong convergence so that the Gibbs states would converge.

Now let us define the class of v's which we are going to study (for motivation see the next section). In the dipole gas case we assume that $e^{-v} \equiv g$ satisfies the following conditions:

(a) $g(\phi)$ is analytic in the strip $|\mathrm{Im}\,\phi| < B$, $g(\phi) = g(-\phi)$, $g(0) = 1$, $|g| \leqslant \exp[\kappa|\phi|^2]$;

(b) for $|\phi| < B$ there exists an analytic v, $v(\phi) = \frac{1}{2}c\phi^2 + \tilde{v}(\phi)$ where $\tilde{v}(0) = 0$, $(d^2\tilde{v}/d\phi^2)(0) = 0$, $|c| < \kappa$, $|\tilde{v}| < \eta$.

There are three parameters entering in the above definition: B, κ, and η. Notice that we admit potentials of rather general type as, e.g., the semibounded polynomials as well as the trigonometric ones (e.g., $\lambda\phi^4$ and $\lambda\cos\phi$). The upper bound in (a) with $\kappa < \frac{1}{2}L^{-d}$ assures stability.

For the ϕ^4 case we take the following assumptions:

(a′) g is analytic in the strip $|\mathrm{Im}\,\phi| < B$, positive for real ϕ, $g(\phi) = g(-\phi)$, $g(0) = 1$, $|g(\phi)| \leqslant \exp[-\frac{1}{2}\eta(\mathrm{Re}\,\phi)^4]$ for $|\phi| \geqslant B$ and $|\mathrm{Im}\,\phi| < 2L^{-1}B$, $\eta > 0$;

(b′) for $|\phi| < B$ there exists an analytic v, $v(\phi) = \frac{1}{2}c\phi^2 + \eta\phi^4 + \tilde{v}(\phi)$ where $\tilde{v}(0) = 0$, $(d^2\tilde{v}/d\phi^2)(0) = 0$, $(d^4\tilde{v}/d\phi^4)(0) = 0$, $|\tilde{v}| \leqslant \eta^2$.

They are again natural since ϕ^4 is the least stable (or marginal for $d = 4$) direction of the linearized transformation dt.

Below we shall state the results obtained in the present paper under the assumptions that $B > B_0$, $\eta < \eta_0$, $L > L_0$, and $\kappa < \kappa_0$ with B_0, η_0, L_0, and κ_0 chosen suitable. The restriction $L \geqslant L_0$ is not serious since we may always increase L by considering a composition of several RG transformations t.

2.1. EXISTENCE OF THE THERMODYNAMICAL LIMIT

The states $\langle\ \rangle_v^N$ given by (4) converge when $N \to \infty$ to infinite volume states $\langle\ \rangle_v$, e.g., in the sense of convergence of expectations of products of ϕ_x's.

2.2. CONVERGENCE OF EFFECTIVE POTENTIALS

A. Dipole Case. $\exp(-t^n v) \to_{n\to\infty} \exp[-\frac{1}{2}c(v)\phi^2]$ uniformly on compacts in $\mathbb{R}$.

B. ϕ^4 Case. There exists $c(v)$ such that for $c = c(v)$ [see (b′) above] $\exp[-t^n v] \to 1$ uniformly on compacts in $\mathbb{R}$ [fixing c to be equal $c(v)$ sets the temperature at its critical value].

2.3. Decay of Correlations

A. Dipole Gas Case. The two-point function satisfies

$$C_1 d(x, y)^{-d} \leqslant |\langle \phi_x \phi_y \rangle_v| \leqslant C_2 d(x, y)^{-d}$$

B. ϕ^4 Case. For $c = c(v)$ (see 2.2B)

$$C_1 d(x, y)^{-d+2} \leqslant |\langle \phi_x \phi_y \rangle_v| \leqslant C_2 d(x, y)^{-d+2}$$

where $d(x, y)$ is defined as $\inf\{L^k : [L^{-k}x] = [L^{-k}y]\}$.

2.4. Analyticity. Dipole Gas Case

If v_λ is an analytic family of potentials satisfying our assumptions (e.g., $v = -\lambda \cos\phi$) then $c(v)$ and the correlation functions are analytic in λ (the same is true about the infinite volume pressure).

3. ITERATIONS OF THE RG

In this section we shall consider the iterations of (2.1). The motivation for the assumptions on v given in Section 2 comes from the following considerations. Take first the $|\psi| < B$-region. If $d\mu$ had compact support (of the order B), the iteration of (2.1) would be easy (see Ref. 2). On the other hand big Z occurs with small $d\mu$ probability. Here comes the assumption (a) or (a′) which allows us to estimate the big Z contribution and to show that it is small. Since the integrand contains only $\gamma\psi$ and $\gamma < 1$ $\exp(-tv)$ will be in fact analytic in a wider strip. This allows us to expand the "small field region" during the iteration, eventually to infinity.

3.1. Dipole Gas

Suppose that g satisfies (a) and (b) with $B = n_0 + n$, $\eta = \delta^{n_0+n}$, $\delta < 1$. $n \geqslant 0$ will count the RG transformations given in terms of g by

$$g'(\phi) = \frac{\int d\mu(z) f(\phi, z)}{\int d\mu(z) f(0, z)} \tag{3.1}$$

where

$$f(\phi, z) = \left[g(L^{-d/2}\phi + z) g(L^{-d/2}\phi - z) \right]^{L^d/2} \tag{3.2}$$

Notice that $f(\phi, z)$ is analytic in ϕ for $|\mathrm{Im}\,\phi| < L^{d/2}B$ and bounded by

$$\begin{aligned} |f(\phi, z)| &\leqslant \exp\left[\tfrac{1}{2} L^d \kappa \left(|L^{-d/2}\phi + z|^2 + |L^{-d/2}\phi - z|^2 \right) \right] \\ &= \exp\left[\kappa \left(|\phi|^2 + L^d |z|^2 \right) \right] \end{aligned} \tag{3.3}$$

Hence $\int d\mu(z) f(\phi, z)$ is analytic for $|\mathrm{Im}\,\phi| < L^{d/2}B$ if $\kappa < \frac{1}{2} L^{-d}$. $\int d\mu(z)$ $f(0, z)$ is easily seen to be nonzero if n_0 is big enough by splitting the z integration to the regions $|z| < B$ and $|z| \geqslant B$ and using (a) and (b). Hence $g'(\phi)$ is analytic for $|\mathrm{Im}\,\phi| < L^{d/2}B$ and

$$|g'(\phi)| \leqslant C \exp\left[\kappa |\phi|^2 \right] \tag{3.4}$$

there.

Now let $|\phi| < (1 - \epsilon) L^{d/2}(n_0 + n + 1)$ and let χ be the characteristic function of the set $\{z : |z| < \frac{1}{2}\epsilon B\}$. Notice that for z in the support of χ $|L^{-d/2}\phi + z| < B$ if n_0 is big enough. Write

$$g'(\phi) = g_1'(\phi)\left[1 + g_2'(\phi) \right] \tag{3.5}$$

where

$$g_1'(\phi) = \frac{\int d\mu(z) \chi(z) f(\phi, z)}{\int d\mu(z) \chi(z) f(0, z)} \tag{3.6}$$

and

$$g_2'(\phi) = \frac{\int d\mu(z)\left[1 - \chi(z) \right] f(\phi, z)}{\left[g_1'(\phi) \int d\mu(z) f(0, z) \right]} \quad - \quad (\phi = 0) \tag{3.7}$$

This makes sense since $g_1'(\phi) \neq 0$ as will be shown below. Notice that

$$g_1'(\phi) = \frac{\exp\left(-\frac{1}{2}c\phi^2\right)\int d\mu(z)\exp\left[-\frac{1}{2}L^d cz^2\right]\chi(z) \times\exp\left\{\frac{1}{2}L^d\left[\tilde{v}\left(L^{-d/2}\phi + z\right) + \tilde{v}\left(L^{-d/2}\phi - z\right)\right]\right\}}{\int d\mu(z)\exp\left(-\frac{1}{2}L^d cz^2\right)\chi(z)\exp\left[L^d\tilde{v}(z)\right]}$$

$$\equiv \exp -\left[\tfrac{1}{2}c\phi^2\right]\tilde{g}_1(\phi)$$

In virtue of (b)

$$|\tilde{g}_1'(\phi) - 1| \leqslant 2L^d\delta^{n_0+n}\exp(C\delta^{n_0+n}) \tag{3.8}$$

Hence

$$\tilde{v}_1' = -\log \tilde{g}_1' \tag{3.9}$$

is analytic and

$$|\tilde{v}_1'| \leqslant 2L^d\delta^{n_0+n}\exp\left[C\delta^{n_0+n}\right] \tag{3.10}$$

Now, since

$$\left|\int d\mu(z)\left[1 - \chi(z)\right]f(\phi,z)\right| \leqslant \exp(\kappa|\phi|^2)\int d\mu(z)\exp(L^d\kappa|z|^2)\left[1 - \chi(z)\right]$$
$$\leqslant \exp\left[-\epsilon'(n_0 + n)^2\right] \tag{3.11}$$

for κ small enough and n_0 big enough

$$|g_2'(\phi)| \leqslant C\exp\left[-\epsilon'(n_0 + n)^2\right] \tag{3.12}$$

Thus

$$\tilde{v}_2' = -\log(1 + \tilde{g}_2') \tag{3.13}$$

is analytic and

$$|\tilde{v}_2'| \leqslant C\exp\left[-\epsilon'(n_0 + n)^2\right] \tag{3.14}$$

We put

$$v' = \tfrac{1}{2}c\phi^2 + \tilde{v}_1' + \tilde{v}_2' \tag{3.15}$$

v' is analytic for $|\phi| < (1-\epsilon)L^{d/2}(n_0 + n + 1)$ and $|\tilde{v}_1 + \tilde{v}_2| \leqslant 2L^d \delta^{n_0+n}\exp(C\delta^{n_0+n})$.

Writing

$$\tilde{v}_1 + \tilde{v}_2 = \tfrac{1}{2}\delta c\phi^2 + \tilde{v}' \tag{3.16}$$

where $(d^2/d\phi^2)\tilde{v}'(0) = 0$ we obtain from the Cauchy formula

$$|\delta c| \leqslant C\delta^{n_0+n} \tag{3.17}$$

Moreover for $|\phi| < n_0 + n + 1$

$$
\begin{aligned}
|\tilde{v}'(\phi)| &\leqslant \sum_{m=4}^{\infty} \frac{1}{m!} \frac{d^m v(0)}{d\phi^m} |\phi|^m \\
&\leqslant \sum_{m=4}^{\infty} 2L^d \delta^{n_0+n} \exp(C\delta^{n_0+n})\left((1-\epsilon)L^{d/2}\right)^{-m} \\
&= 2(1-\epsilon)^{-4} L^{-d} \delta^{n_0+n} \exp(C\delta^{n_0+n}) \Big/ \left[1 - (1-\epsilon)^{-1} L^{-d/2}\right] \\
&\leqslant \delta^{n_0+n+1}
\end{aligned}
\tag{3.18}
$$

if ϵ, δ are close to 1, $L \geqslant L_0$ and n_0 is big enough. Thus we see that for $|\phi| < n_0 + n + 1$ $g'(\phi) = \exp[-v'(\phi)]$ where $v'(\phi)$ is analytic, $v'(\phi) = \frac{1}{2}c'\phi^2 + \tilde{v}'(\phi)$, $c' = c + \delta c$ with $|\delta c| \leqslant \delta^{n_0+n}$, $\tilde{v}'(0) = 0$, $(d^2\tilde{v}'/d\phi^2)(0) = 0$ and $|\tilde{v}'| \leqslant \delta^{n_0+n+1}$. This implies that for $|\phi| < n_0 + n + 1$

$$|g'(\phi)| \leqslant \exp(\kappa|\phi|^2) \tag{3.19}$$

Since by (4) for $|\phi| \geqslant n_0 + n + 1$, $|\mathrm{Im}\,\phi| < n_0 + n + 1$

$$|g'(\phi)| \leqslant \exp\left\{\left[\kappa + C(n_0+n)^{-2}\right]|\phi|^2\right\} \tag{3.20}$$

we finally conclude that (a), (b) for g with $B = n_0 + n$, $\eta = \delta^{n_0+n}$, δ close to 1, κ small and n_0 sufficiently large yields (a), (b) for g' with $B = n_0 + n + 1$, $\eta = \delta^{n_0+n+1}$, κ increased by $C(n_0+n)^{-2}$ and $|c' - c| \leqslant \delta^{n_0+n}$. Iteration of this result proves the convergence of the effective potentials in the dipole case claimed in Section 2.

3.2. ϕ_d^4

We shall only consider here the $d = 4$ case, which is more subtle than the other ones. Consider, e.g., $v(\phi) = \lambda\phi^4$. To gain some insight, we compute perturbatively in λ

$$tv = (6L^2\lambda - 72L^6\lambda^2)\phi^2 + (\lambda - 36L^4\lambda^2)\phi^4 + 0(\lambda^3) + 0(\phi^6) \tag{3.21}$$

Thus to the leading order ϕ^4 is marginal. If we go to the second order we get

$$\lambda_{n+1} = \lambda_n - 36L^4\lambda_n^2 \tag{3.22}$$

the solution of which behaves as $[36L^4(n_0+n)]^{-1}$ when $n \to \infty$, which is the familiar logarithmic approach to zero. This forces us to be very careful about the n-dependence of the constants during the iteration process.

We shall adapt the assumptions (a′) and (b′) to the inductive procedure taking $B = (n_0 + n)^\alpha$ with α equal, e.g., to $\frac{1}{10}$ and η satisfying

$$\frac{C_-}{n_0 + n} \leqslant \eta \leqslant \frac{C_+}{n_0 + n}$$

where $C_+^{-1} < 36L^4 < C_-^{-1}$. We shall assume that c is a continuous function of a parameter c_0 (e.g., a half of the coefficient at ϕ^2 in the initial v) such that when c_0 sweeps an interval $[\alpha_n, \beta_n]$ then $c(c_0)$ goes from $-\bar{c}/(n_0 + n)$ to $\bar{c}/(n_0 + n)$.

We shall show that if constants are chosen properly then (a′), (b′) with given n for g implies (a′), (b′) with $n + 1$ and $[\alpha_{n+1}, \beta_{n+1}] \subset [\alpha_n, \beta_n]$ for g'. Iteration of this result proves the convergence of the effective potentials for the ϕ_4^4 case claimed in Section 2. $c(v)$ is given by the point in the intersection of all $[\alpha_n, \beta_n]$.

Let now

$$f(\phi, z) = \left[g(L^{-1}\phi + z) g(L^{-1}\phi - z) \right]^{L^4/2} \tag{3.23}$$

Notice that $f(\phi, z)$ is analytic for $|\mathrm{Im}\,\phi| < L(n_0 + n)^\alpha$ and uniformly bounded for $|\mathrm{Im}\,\phi| < 2(n_0 + n)^\alpha$. Thus $g'(\phi)$ is an analytic function of ϕ for $|\mathrm{Im}\,\phi| < 2(n_0 + n)^\alpha$. As in the dipole case write for $|\phi| < (n_0 + n + 1)^\alpha$

$$g'(\phi) = g_1'(\phi)\left[1 + g_2'(\phi) \right]$$

where g_1', g_2' are given by (3.6), (3.7) with χ being the characteristic function of $\{z : |z| < \frac{1}{2}\epsilon(n_0 + n)^\alpha\}$. Notice that $g_1'(\phi)$ is analytic for $|\phi| < (1 - \epsilon)L(n_0 + n + 1)^\alpha$ and

$$|g_1'(\phi) - 1| \leqslant C(n_0 + n)^{-1 + 4\alpha} \tag{3.24}$$

Thus for $|\phi| < (1 - \epsilon)L(n_0 + n + 1)^\alpha$ and n_0 big enough

$$v_1' = -\log g_1' \tag{3.25}$$

is analytic. We have to analyze v_1' to the second order in the perturbation expansion:

$$\begin{aligned} v_1' = {} & \tfrac{1}{2}cL^2\phi^2 + \eta\phi^4 - \log \int d\mu(z)\chi(z)\exp\left(-\tfrac{1}{2}cL^4z^2 - \eta L^4 z^4\right) \\ & \times \tilde{f}(\phi, z)\exp\left[-6\eta L^2\phi^2 z^2\right] \\ & + \log \int d\mu(z)\chi(z)\exp\left[-\tfrac{1}{2}cL^4z^2 - \eta L^4z^4\right]\tilde{f}(0, z) \end{aligned} \tag{3.26}$$

where $\tilde{f}$ is given by (3.23) with g replaced by $\exp(-\tilde{v})$. Introducing

$$\langle - \rangle_\phi = \frac{1}{\mathcal{N}} \int d\mu(z)\chi(z)\exp\left(-\tfrac{1}{2}cL^4z^2 - \lambda L^4z^4\right)\tilde{f}(\phi, z) - \tag{3.27}$$

and

$$\tilde{g}_1'(\phi) = \frac{\int d\mu(z)\chi(z)\exp\left[-\frac{1}{2}cL^4z^2 - \eta L^4 z^4\right]\tilde{f}(\phi, z)}{\int d\mu(z)\chi(z)\exp\left(-\frac{1}{2}cL^4z^2 - \eta L^4 z^4\right)\tilde{f}(0, z)} \tag{3.28}$$

we obtain

$$v_1' = \tfrac{1}{2}cL^2\phi^2 + \eta\phi^4 - \log\left\langle \exp(-6\eta L^2\phi^2 z^2)\right\rangle_\phi - \log \tilde{g}_1' \tag{3.29}$$

Notice that for n_0 big enough

$$|-\log \tilde{g}_1'| \leqslant 2L^4\kappa^{-1}\eta^2 e^{C\kappa^{-1}\eta^2} \tag{3.30}$$

Now

$$-\log\left\langle \exp\left[-6\eta L^2\phi^2 z^2\right]\right\rangle_\phi = 6\eta L^2\phi^2\langle z^2\rangle_\phi - 18\eta^2L^4\phi^4\langle z^2; z^2\rangle_\phi^T + O(\eta^3\phi^6) \tag{3.31}$$

Moreover

$$\langle z^2\rangle_\phi = 1 + O\left((n_0+n)^{-1}\right) + O\left((n_0+n)^{-2}\right) \tag{3.32}$$

with first order ϕ-independent,

$$\langle z^2; z^2\rangle_\phi^T = 2 + O\left((n_0+n)^{-1}\right) \tag{3.33}$$

Using the Cauchy integral formula to estimate the derivatives we obtain

$$v_1' = \tfrac{1}{2}c_1'\phi^2 + \eta_1'\phi^4 + \tilde{v}_1' \tag{3.34}$$

where

$$\tilde{v}_1'(0) = \frac{d^2\tilde{v}_1'(0)}{d\phi^2} = \frac{d^4\tilde{v}_1'(0)}{d\phi^4} = 0$$

and

$$c_1' = (c + 6\eta)L^2 + O\left((n_0+n)^{-2}\right) \tag{3.35}$$

$$\eta_1' = \eta - 36L^4\eta^2 + O\left((n_0+n)^{-2-4\alpha}\right) \tag{3.36}$$

$$|\tilde{v}_1| \leqslant \delta\kappa^{-1}\eta^2 \tag{3.37}$$

provided that we restrict ourselves to $|\phi| < (n_0 + n + 1)^\alpha$.

As far as the g_2' term is concerned it is a small correction,

$$|g_2'| \leqslant \exp\left[-\epsilon'(n_0+n)^{2\alpha}\right] \tag{3.38}$$

so that

$$v' = v_1' - \log(1 + g_2') = \tfrac{1}{2}c'\phi^2 + \eta'\phi^4 + \tilde{v} \tag{3.39}$$

where c', η', and $\tilde{v}'$ satisfy (3.35)–(3.37). When c sweeps the interval

$$\left[\frac{-\bar{c}}{n_0+n}, \frac{\bar{c}}{n_0+n}\right]$$

c' covers an interval

$$\left[\frac{(-\bar{c}+C_+)L^2}{n_0+n} + O\big((n_0+n)^{-2}\big), \frac{(\bar{c}+C_-)L^2}{n_0+n} - O\big((n_0+n)^{-2}\big)\right]$$

which contains

$$\left[\frac{-\bar{c}}{n_0+n+1}, \frac{\bar{c}}{n_0+n+1}\right]$$

if, say, $\bar{c} \geqslant 2C_+$ and n_0 is big enough. Hence the existence of $[\alpha_{n+1}, \beta_{n+1}] \subset [\alpha_n, \beta_n]$ with required properties follows. (36) and (37) imply clearly that

$$|\tilde{v}_1| \leqslant \kappa^{-1}\eta'^2 \tag{3.40}$$

We shall show now that

$$\frac{C_-}{n_0+n+1} \leqslant \eta' \leqslant \frac{C_+}{n_0+n+1} \tag{3.41}$$

Notice that

$$\eta'^{-1} = \eta^{-1}\left[1 - 36L^4\eta + O((n_0+n))^{-1-4\alpha}\right]^{-1} \tag{3.42}$$

Hence

$$\eta^{-1}\left[1 + (36L^4 - \epsilon)\eta\right] \leqslant \eta'^{-1} \leqslant \eta^{-1}\left[1 + (36L^4 + \epsilon)\eta\right] \tag{3.43}$$

where $\epsilon \to 0$ if $n_0 \to \infty$. Thus

$$C_+^{-1}(n_0+n) + 36L^4 - \epsilon \leqslant \eta'^{-1} \leqslant C_-^{-1}(n_0+n) + 36L^4 + \epsilon \tag{3.44}$$

and consequently

$$C_+^{-1}(n_0+n+1) \leqslant \eta'^{-1} \leqslant C_-^{-1}(n_0+n+1)$$

which is (3.41).

We are left with bounding $g'(\phi)$ for $|\phi| \geqslant (n_0+n+1)^\alpha$, $|\operatorname{Im}\phi| \leqslant 2L^{-1}(n_0+n+1)^\alpha$. Write $\phi = \phi^1 + i\phi^2$ and $\psi_\pm = L^{-1}\phi \pm z$. Then we have

$$g'(\phi) = \frac{1}{\mathfrak{N}}\left(\int_{|z| \leqslant L^{-1}|\phi_1|/2} d\mu(z)\left[g(\psi_+)g(\psi_-)\right]^{L^4/2} + \int_{|z| \geqslant L^{-1}|\phi_1|/2} d\mu(z)\left[g(\psi_+)g(\psi_-)\right]^{L^4/2}\right) \tag{3.45}$$

We claim that for $|z| \leqslant \frac{1}{2} L^{-1}|\phi_1|$

$$|g(\psi_\pm)| \leqslant \exp\left[-\tfrac{1}{2}\eta(\mathrm{Re}\,\psi_\pm)^4\right] \tag{3.46}$$

Indeed. If $|\psi_\pm| \geqslant (n_0+n)^\alpha$ then (3.46) holds by (a). But $|\psi_\pm| \geqslant L^{-1}|\phi| - |z| \geqslant \frac{1}{2}L^{-1}|\phi| \geqslant \frac{1}{2}L^{-1}(n_0+n+1)^\alpha$ and $|\mathrm{Im}\,\psi_\pm| = L^{-1}|\phi_2| < 2L^{-2}(n_0+n+1)^\alpha$. Hence by (b) for $|\psi_\pm| < (n_0+n)^\alpha$ Equation (3.46) holds too if L is big enough so that $\psi_\pm$ has a small argument. Equation (3.46) implies

$$\begin{aligned}
&\left|\int_{|z|<L^{-1}|\phi_1|/2} d\mu(z)\left[g(\psi_+)g(\psi_-)\right]^{L^4/2}\right| \\
&\quad \leqslant \int_{|z|<L^{-1}|\phi_1|/2} d\mu(z)\exp\left\{-\tfrac{1}{4}\eta L^4\left[\left(L^{-1}\phi_1+z\right)^4\right]+\left(L^{-1}\phi_1-z\right)^4\right\} \\
&\quad \leqslant \exp\left(-\tfrac{1}{2}\eta\phi_1^4\right)\int d\mu(z)\exp\left[-\tfrac{3}{2}\eta L^2\phi_1^2 z^2\right] \\
&\quad \leqslant \exp\left[-\tfrac{1}{2}\eta\phi_1^4-\epsilon(n_0+n)^{-1+2\alpha}\right]
\end{aligned} \tag{3.47}$$

As far as the second term of (3.45) is concerned, notice that if $|z| \geqslant \frac{1}{2}L^{-1}|\phi_1|$ then either $|\mathrm{Re}\,\psi_+| \geqslant \frac{3}{2}L^{-1}|\phi_1|$ or $|\mathrm{Re}\,\psi_-| \geqslant \frac{3}{2}L^{-1}|\phi_1|$. Thus $|g(\psi_+)g(\psi_-)| \leqslant 2\exp(-\eta L^{-4}|\phi_1|^4)$, say, and

$$\begin{aligned}
&\left|\int_{|z|\geqslant L^{-1}|\phi_1|/2} d\mu(z)\left[g(\psi_+)g(\psi_-)\right]^{L^4/2}\right| \\
&\quad \leqslant C\exp\left[-\tfrac{1}{2}\eta|\phi_1|^4\right]\int_{|z|\geqslant L^{-1}|\phi_1|/2} d\mu(z) \\
&\quad \leqslant \exp\left[-\tfrac{1}{2}\eta|\phi_1|^4-\epsilon(n_0+n)^{2\alpha}\right]
\end{aligned} \tag{3.48}$$

$$|\mathfrak{N}| = \left|\int d\mu(z)g(z)^{L^4}\right| \geqslant \exp\left[-C(n_0+n)^{-1}\right] \tag{3.49}$$

(3.45), (3.47)–(3.49) imply

$$|g'(\phi)| \leqslant \exp -\left[\tfrac{1}{2}\eta(\mathrm{Re}\,\phi)^4\right]$$

for $|\phi| \geqslant (n_0+n+1)^\alpha$, $|\mathrm{Im}\,\phi| < 2L^{-1}(n_0+n+1)^\alpha$. This ends the induction step.

4. THE CORRELATIONS

The analysis of the correlation functions is an elaboration on the ideas already presented. We shall therefore be brief and shall only do in detail the two-print function of ϕ_4^4. The $d>4$ cases and the dipole gas are even easier and we leave them as an exercise.

The two-point function is $\langle\phi_x\phi_y\rangle_v$ (we suppress N). Let $[L^{-1}x]$

$\neq [L^{-1}y]$. In this case the Z^0 integral factors and since

$$\int d\mu(z) f(\phi, z) z = 0$$

we get

$$\langle \phi_x \phi_y \rangle_v = L^{-2} \langle \phi_{[L^{-1}x]} \phi_{[L^{-1}y]} \rangle_{tv} \tag{4.1}$$

Let k be the first integer such that $[L^{-k}x] = [L^{-k}y] \equiv z$. Iteration of (4.1) gives

$$\langle \phi_x \phi_y \rangle = L^{-2(k-1)} \langle \phi_{[L^{-k+1}x]} \phi_{[L^{-k+1}y]} \rangle_{v_{k-1}} = \langle G_k(\phi_z)\, g_k(\phi_z)^{-1} \rangle_{v_k} \tag{4.2}$$

where in the second step we did one more z integration and defined

$$G_k(\phi) = \left(L^{-2k}\phi^2 + L^{-2(k-1)} A_{[L^{-k+1}x]} A_{[L^{-k+1}y]} \langle z^2 \rangle_{k-1} \right) g_k(\phi) \tag{4.3}$$

with

$$v_k \equiv t^k v, \; g_k \equiv e^{-v_k}$$

$$\langle z^2 \rangle_{k-1} \equiv \frac{\int d\mu(z) f_{k-1}(\phi, z) z^2}{\int d\mu(z) f_{k-1}(\phi, z)} \tag{4.4}$$

$$f_n(\phi, z) = \left[g_n(L^{-1}\phi + z) g_n(L^{-1}\phi - z) \right]^{L^4/2} \tag{4.5}$$

Subsequent integrations give

$$\langle \phi_x \phi_y \rangle_v = \left\langle G_{k+1}(\phi_{[L^{-1}z]}) g_{k+1}(\phi_{[L^{-1}z]})^{-1} \right\rangle_{v_{k+1}} \tag{4.6}$$

where for $l \geqslant 1$

$$G_{k+1}(\phi) = \frac{\begin{array}{c}\int d\mu(z) G_{k+l-1}(L^{-1}\phi + z)\left[g_{k+l-1}(L^{-1}\phi + z) \right]^{L^4 - 1/2} \\ \times \left(g_{k+l-1}(L^{-1}\phi - z) \right)^{L^4/2}\end{array}}{\int d\mu(z)_{k+l-1}(0, z)} \tag{4.7}$$

With due assumptions v_n behaves as discussed in the previous section. Now the claim is that G_n's have the following properties which will be proven inductively:

(A_n) G_n is analytic for $|\mathrm{Im}\,\phi| < (n_0 + n)^\alpha$, $G_n(\phi) = G_n(-\phi)$ and $\kappa^{-1} |G_n| \leqslant (n_0 + n)^{2\alpha} \exp[-\frac{1}{3}\eta(\mathrm{Re}\,\phi)^4]$ for $|\phi| \geqslant (n_0 + n)^\alpha$, $|\mathrm{Im}\,\phi| < 2L^{-1}(n_0 + n)^\alpha$,

(B_n) for $|\phi| < (n_0 + n)^\alpha$

$$F_n(\phi) \equiv G_n(\phi)\, g_n(\phi)^{-1} = b_{n0} + b_{n2}\phi^2 \tilde{F}_n(\phi)$$

where $\tilde{F}_n(0) = (d^2/d\phi^2)\tilde{F}_n(0) = 0$ and $\kappa|\tilde{F}_n| \leqslant |b_{n2}|(n_0 + n)^{-1/2}$

(C_n) $\kappa|b_{n0} - b_{(n-1)0} - b_{(n-1)2}| \leqslant 2|b_{(n-1)2}|(n_0 + n)^{-1/2}$
$\kappa|b_{n2} - L^{-2}b_{(n-1)2}| \leqslant |b_{(n-1)2}|(n_0 + n)^{-1/2}$

Let us consider G_k first as given by (4.3)–(4.5). The properties (A_k) are proven as (a′) in Section 3. Since

$$F_k = L^{-2k}\phi^2 + L^{-2(k-1)}A_{[L^{-k+1}x]}A_{[L^{-k+1}y]}\langle z\rangle_{k-1}^2 \tag{4.8}$$

and

$$\langle z^2\rangle_{k-1}\restriction_{\phi=0} = 1 + 0\left((n_0+k)^{-1}\right) \tag{4.9}$$

we have

$$b_{k0} = \pm L^{2(k-1)}\left[1 + 0\left((n_0+k)^{-1}\right)\right] \tag{4.10}$$

Now a straightforward estimation shows that

$$|L^{-2(k-1)}A.A.\langle z^2\rangle_{k-1} - b_{k0}| \leqslant L^{-2(k-1)}O\left((n_0+k)^{-1+2\alpha}\right) \tag{4.11}$$

Equation (11) implies via the Cauchy integral formula that

$$|b_{k2} - L^{-2k}| \leqslant L^{-2(k-1)}O\left((n_0+k)^{-1}\right) \tag{4.12}$$

$$|\tilde{F}_k| \leqslant L^{-2(k-1)}O\left((n_0+k)^{-1+2\alpha}\right) \tag{4.13}$$

which imply in turn (B_k).

We shall pass now to a general induction step. Assume we have shown (A_n)–(C_n) for $n \leqslant k+l-1$. (A_{k+l}) follows again as (a′) of Section 3. For $|\phi| < (n_0+k+l)^\alpha$ we may rewrite (4.7) separating the small and large z integrations:

$$F_{k+l}(\phi) = \frac{\int d\mu(z)\chi(z)F_{k+l-1}(L^{-1}\phi+z)f_{k+l-1}(\phi,z)}{\int d\mu(z)f_{k+l-1}(\phi,z)}$$
$$+ \frac{\begin{array}{c}\int d\mu(z)\left[1-\chi(z)\right]G_{k+l-1}(L^{-1}\phi+z)\\ \times\left[g_{k+l-1}(L^{-1}\phi+z)\right]^{L^4/2-1}\left[g_{k+l-1}(L^{-1}\phi-z)\right]^{L^4/2}\end{array}}{\int d\mu(z)f_{k+l-1}(0,z)} \tag{4.14}$$

where, as before, $\chi(z)$ is the characteristic function of the set $\{z : |z| < \frac{1}{2}\epsilon(n_0+k+l-1)^\alpha\}$. The second term is easily shown to be bounded by $\exp[-\epsilon'(n_0+k+l)^{2\alpha}]$. The first one is equal to [see (4.4) for the notation]

$$\begin{aligned}&\left\langle\chi(z)F_{k+l-1}(L^{-1}\phi+z)\right\rangle_{k+l-1}\\ &\quad= \left(b_{(k+l-1)0} + b_{(k+l-1)2}L^{-2}\phi^2\right)\\ &\qquad\times\left\{1 + O\left(\exp\left[-\epsilon'(n_0+k+l)^{2\alpha}\right]\right)\right\} + b_{(k+l-1)2}\left\langle\chi(z)z^2\right\rangle_{k+l-1}\\ &\qquad+\left\langle\chi(z)\tilde{F}_{k+l-1}(L^{-1}\phi+z)\right\rangle_{k+l-1}\end{aligned} \tag{4.15}$$

Since

$$b_{(k+l-1)2}\left\langle\chi(z)z^2\right\rangle_{k+l-1}\restriction_{\phi=0}=b_{(k+l-1)2}\left[1+0\left((n_C+k+l)^{-1}\right)\right] \quad (4.16)$$

and

$$\left|\left\langle\chi(z)\tilde{F}_{k+l-1}(L^{-1}\phi+z)\right\rangle_{k+l-1}\right|\leqslant\kappa^{-1}|b_{(k+l-1)2}|(n_0+k+l-1)^{-1/2} \quad (4.17)$$

(4.14) and (4.15) imply the first inequality of (C_{k+l}). The second inequality also follows, via the Cauchy estimate, from (4.14), (4.15), (4.17) and

$$\begin{aligned}&\left|b_{(k+l-1)}\left(\left\langle\chi(z)z^2\right\rangle_{k+l-1}-\left\langle\chi(z)z^2\right\rangle_{k+l-1}\restriction_{\phi=0}\right)\right|\\&\quad\leqslant b_{(k+l-1)2}O\left((n_0+k+l)^{-1+2\alpha}\right)\end{aligned} \quad (4.18)$$

Moreover the contribution to $\tilde{F}_{k+l}$ from $b_{(k+l-1)2}\langle\chi(z)z^2\rangle_{k+l-1}$ is bounded by $b_{(k+l-1)2}O((n_0+k+l)^{-1+2\alpha})$. The other one, from $\langle\chi(x)\tilde{F}_{k+l-1}(L^{-1}\phi+z)\rangle_{k+l-1}$ is equal to the contribution from

$$\left[\int d\mu(z)\chi(z)\tilde{F}_{k+l-1}(L^{-1}\phi+z)f_{k+l-1}(\phi,z)\Big/\int d\mu(z)\chi(z)f_{k+l-1}(\phi,z)\right]$$

for a term bounded by $\kappa^{-1}|b_{(k+l-1)2}|\exp[-\epsilon(n_0+k+l)^{2\alpha}]$. $[\cdots]$ is analytic even for $|\phi|<(1-\epsilon)L(n_0+k+l)^\alpha$ and bounded there by $\kappa^{-1}|b_{(k+l-1)2}|(n_0+k+l-1)^{-1/2}$. Its contribution to $\tilde{F}_{k+l}$ is estimated for $|\phi|<(n_0+k+l)^\alpha$ by

$$\begin{aligned}&\sum_{m=4}^{\infty}\frac{1}{m!}\left|\frac{d^m}{d\phi^m}\right|_{\phi=0}[\cdots]|\phi|^m\\&\quad\leqslant(1-\epsilon)^{-4}L^{-4}\left[1-(1-\epsilon)^{-1}L^{-1}\right]^{-1}\\&\quad\quad\times\kappa^{-1}|b_{(k+l-1)2}|(n_0+k+l-1)^{-1/2}\\&\quad\leqslant\tfrac{1}{2}\kappa^{-1}|L^{-2}b_{(k+l-1)2}|(n_0+k+l)^{-1/2}\end{aligned} \quad (4.19)$$

for $L\geqslant L_0$ and ϵ small (we have used again the Cauchy estimate for the derivatives). Hence (B_{k+l}) follows.

From (C_n) we easily obtain for $n\geqslant k+1$

$$\left.\begin{aligned}&b_{(n-1)2}\\&b_{n0}-b_{(n-1)0}\end{aligned}\right\}=L^{-2(n-1)}\exp\left[0\left((n_0+k)^{-1/2}\right)(n-k)\right] \quad (4.20)$$

Since

$$\langle \phi_x \phi_y \rangle_v = b_{(N+1)0} = b_{k0} + \sum_{n=k+1}^{N+1} \left(b_{n0} - b_{(n-1)0} \right)$$
$$= L^{2(k-1)} \left[\pm 1 + \frac{1}{L^2 - 1} + O\left((n_0 + k)^{-1/2} \right) + O\left(L^{-2(N-k)} \right) \right] \tag{4.21}$$

the existence of the thermodynamic limit $N \to \infty$ for the two-point function follows immediately as well as the decay as $d(x, y)^{-2}$ claimed in Section 2.

For general (even) correlation functions $\langle \prod_{i=1}^{2m} \phi_{x_i} \rangle_v$ we shall integrate out $Z^0, Z^1, \ldots$ up to Z^k, where k is the smallest integer such that all $[L^{-k} x_i]$ are equal. This gives

$$\left\langle \prod_{i=1}^{2m} \phi_{x_i} \right\rangle = \left\langle \tilde{G}_k(\phi) g_k(\phi)^{-1} \right\rangle_{v_k} \tag{4.22}$$

$\tilde{G}_k$ is easily proven to be analytic for $|\mathrm{Im}\,\phi| < (n_0 + k)^\alpha$, $\tilde{G}_k(\phi) = \tilde{G}_k(-\phi)$, $\kappa^{-1} |\tilde{G}_k| \leqslant (n_0 + k)^{m\alpha} \exp[-\frac{1}{3}\eta(\mathrm{Re}\,\phi)^4]$. Further Z integrations are done as in the case of the two-point function and the existence of the thermodynamic limit follows as before. It should be clear that a closer analysis of $\tilde{G}_k$ would allow to control the decay of the $2m$-point function. This is left to the reader.

Finally the analyticity results stated for the dipole gas case follow in a straightforward way from the analysis of Sections 3, 4 and the Vitali theorem.

ACKNOWLEDGMENTS

The authors wish to thank Professors J. Fröhlich and N. Kuiper for inviting them to IHES, where most of this work was carried out.

REFERENCES

1. K. Gawędzki and A. Kupiainen, *Commun. Math. Phys.* **77**:31–64 (1980).
2. K. Gawędzki and A. Kupiainen, *Commun. Math. Phys.* **82**:407–433 (1981).
3. K. Gawędzki and A. Kupiainen, *Commun. Math. Phys.* **83**:469–492 (1982).
4. F. J. Dyson, *Commun. Math. Phys.* **12**:91–107 (1969); and **21**:269–283 (1971).
5. P. M. Bleher and Ya. G. Sinai, *Commun. Math. Phys.* **33**:23–42 (1973).
6. Ya. G. Sinai, *Theory of Phase Transitions* (in Russian) (Nauka, Moscow, 1980).
7. J. Fröhlich, *Nucl. Phys. B* **200**[FS4]:281–296 (1982).
8. K. Gawędzki and A. Kupiainen, "Renormalization group for a critical lattice model. Effective iterations beyond the perturbation expansion or bounded spin approximation," preprint IHES/P/82/15, to appear in *Commun. Math. Phys.*